Aggregation–Diffusion Equations: concentration vs simplification

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Aggregation and Diffusion

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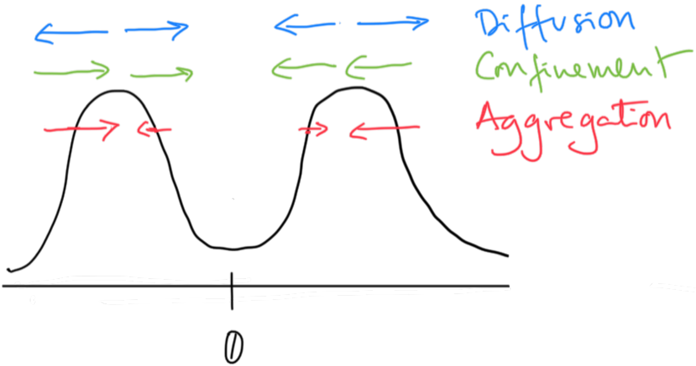

A model of diffusion, confinement and aggregation

The aim of this talk is to explain the modeling and theory behind the following model for aggregation-diffusion phenomena:

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla \big(\underbrace{U'(\rho)}_{\text{Diffusion}} + \underbrace{V}_{\text{Confinement}} + \underbrace{W * \rho}_{\text{Aggregation}} \big)\right) \tag{ADE}$$

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We will discuss the range of power-type aggregation and diffusion

$$U'(\rho) = \tfrac{m}{m-1}\rho^{m-1}, \qquad V(x) = \frac{|x|^\alpha}{\alpha}, \qquad \text{and} \qquad W(x) = \frac{|x|^\lambda}{\lambda}.$$

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If V, W are bounded below, we can always assume $V, W \geq 0$.

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Conservation equation. Let ρ be a density and $\omega \subset \mathbb{R}^d$ any control volume, if $\mathbf{j}$ is the out-going flux

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \rho \,\mathrm{d}x = -\int_{\partial\omega} \mathbf{j}\cdot\mathbf{n}\,\mathrm{d}S = -\int_\omega \operatorname{div}\mathbf{j}\,\mathrm{d}x$$

Conservation equation. The heat equation

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Notice $\Delta \varphi(\rho) = \operatorname{div}(\varphi'(\rho) \nabla \rho)$ so $\qquad U''(\rho) = \frac{\varphi'(\rho)}{\rho}$.

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[1] Assume $\nabla W(0) = 0$

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Consider N with positions X_i of equal masses $1/N$ and the attracting/repulsive system[1]

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Diffusion can added to the particle system by introducing noise ▸ Details .

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Keller-Segel ($d = 2$)	$\rho \log \rho$	0	$-\frac{1}{2\pi}\log \lvert x\rvert$
Swarming / Herding	0	0	$\frac{1}{a}\lvert x\rvert^a - \frac{1}{b}\lvert x\rvert^b$

Conservation?

In conservation laws, we expect $\int_{\mathbb{R}^d} \rho(t) = \int_{\mathbb{R}^d} \rho_0$

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Sometimes mass is not conserved, and we will give an example later.

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- Any solution satisfies

$$\|\rho(t,\cdot) - K(t,\cdot)\|_{L^1} \to 0 \qquad \text{as } t \to \infty.$$

This is known as **asymptotic simplication**.

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Diffusion phenomena: the Porous Medium Equation: $\partial_t \rho = \Delta \rho^m$

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- There is **finite-time extinction**

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- [Brezis and Friedman 1983] proved that δ_0 does not diffuse:
  if we take a sequence $\rho_j(0^+, \cdot) \to \delta_0$, the associated solution $\rho_j(t) \to \delta_0$.

Notice that the heat kernel is **self-similar**

$$K(t,x) = (2t)^{-\frac{d}{2}} G(-\frac{x}{\sqrt{2t}}), \qquad G(y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \exp(-\frac{|y|^2}{2})$$

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A similar approach works for the Porous Medium Equation, where the profile is B.

The Keller-Segel proposed a model of cell migration by chemotaxis given by

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There exists $M^* > 0$ such that

- If $M < M^*$ solutions are global-in-time.
- If $M > M^*$ there is finite-time blow-up. And $\rho(T^*) = M\delta_0$.

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Finite time blow-up

Mathematical Institute

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  When $M > M^*$ there exist $\|\rho_0\|_{L^1} = M$ such that $\mathcal{F}[\rho_0] < 0$.

Main question

For

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla \left(U'(\rho) + V + W * \rho\right)\right) \tag{ADE}$$

can we classify characterise ρ_∞ such that

$$\rho(t) \to \rho_\infty$$

in terms of general U, V, W?

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Let $F : \mathbb{R}^d \to \mathbb{R}$. Imagine we look for $\operatorname{argmin} F$.

Gradient flow in $\mathbb{R}^d$

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If $D^2 F \geq \lambda I$ then $|X(t) - X_\infty| \leq e^{-\lambda t}|X_0 - X_\infty|$.

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In general, the $\nabla_{L^2} \mathcal{F}$ is given by the Euler-Lagrange equations ▸ Details

Our equations are "nice" in 2-Wasserstein space ($\mathcal{P}_2$). ▸ Details

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If $W(x) = W(-x)$, we can formally rewrite the Aggregation-Diffusion problem as[2]

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[2] Due to the convolution, $\mathcal{F}$ is non-local and $\mathcal{F}[\rho] \neq \int_{\mathbb{R}^d} F(x, \rho(x))\, \mathrm{d}x$. $\frac{\delta F}{\delta \rho}$ can be computed directly

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The precise definition of solution is the notion of *curves of maximal slope*. ▸ Details

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Due to the estimate above, at an minimiser ρ_∞, we have

$$\rho_\infty \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = 0.$$

Either $\rho_\infty = 0$ (as in PME), or $\frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = C$ (over open sets).

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In some cases it is displacement convex (see [Carrillo and Slepčev 2009]): $m \geq \frac{d-1}{d}$ and V, W convex.

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The first variation is: $\dfrac{\delta \mathcal{F}}{\delta \rho} = U'(\rho) + V + W * \rho.$

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This gives the intuition that for $M > 8\pi$ then δ_0 (i.e. $\lambda \to \infty$) is energy beneficial.

Free-energy minimisation for (ADE) when $V = 0$

Existence and non-existence of δ_0

Minimisation for $U = \frac{m}{m-1}\rho^m$, $V = 0$, and $W(x) = |x|^\lambda/\lambda$:

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The question is

$$\rho(t) \to \rho_\infty ?$$

Outside the displacement convex range, we have to go back to PDE methods.

[Cao and Li 2020]: If $\rho_{V+h_1} \leq \rho_0 \leq \rho_{V+h_2}$ with $h_1, h_2 > 0$ then L^∞_{loc} convergence to

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Some positive answers

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Then $\exists \rho_0 \in L^1(\mathbb{R}^d)$ such that $\forall r > 0$

$$M(t,r) = \int_{B_r} \rho(t,x)\,\mathrm{d}x \nearrow \int_{B_r} \rho_V(x)\,\mathrm{d}x + \underbrace{1 - \|\rho_V\|_{L^1}}_{\text{a Dirac!}} \qquad \text{with } a > 0 \text{ as } t \to \infty.$$

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The key idea is that M satisfies a Hamilton-Jacobi type equation.

▸ Sketch of Proof

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We start by a classical observation.

Remark

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If $W \in L^\infty(\mathbb{R}^d)$, then there are no finite-mass steady states.

If $W \in L^\infty(\mathbb{R}^d)$ and $\rho \in L^1(\mathbb{R}^d)$, then $W * \rho \in L^\infty(\mathbb{R}^d)$.
The Euler-Lagrange equation is $\log \rho + W * \rho = c$, so

$$\rho = e^c e^{W*\rho} \geq e^c e^{-\|W*\rho\|_{L^\infty}} > 0.$$

Therefore, in this range we always expect diffusion.

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Theorem [Carrillo, G-C, Yao, and Zeng 2021] ▸ Sketch of proof

Let $n \geq 2$, and assume $W(x) = W(-x)$

- $W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$
- $\nabla W \in L^{n-\varepsilon}(\mathbb{R}^d)$
- $\Delta W \in L^{\frac{n}{2}}(\mathbb{R}^d)$ (and also $\Delta W \in L^{\frac{n}{2}-\varepsilon}(\mathbb{R}^d)$ if $n \geq 3$)

Then $(\star)$.

Notice that this hypothesis work for $W(x) \sim |x|^{-\varepsilon}$ for any $\varepsilon > 0$,
but not for the critical case $W(x) \sim \log |x|$.

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The answer seems positive for some W.
PhD plan of Alejandro Fernández-Jiménez (U. Oxford).

Thank you for your attention!

L. Ambrosio, N. Gigli, and G. Savare. *Gradient Flows*. Lectures in Mathematics ETH Zürich. Basel: Birkhäuser-Verlag, 2005, pp. 1–27.

A. Blanchet, J. A. Carrillo, and P. Laurençot. "Critical mass for a Patlak–Keller–Segel model with degenerate diffusion in higher dimensions". *Calculus of Variations and Partial Differential Equations* 35.2 (Aug. 2009), pp. 133–168.

A. Blanchet, J. Dolbeault, and B. Perthame. "Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions.". *Electronic Journal of Differential Equations (EJDE)[electronic only]* (2006).

H. Brezis and A. Friedman. "Nonlinear parabolic equations involving measures as initial data". *J. Math. Pures Appl.* 62 (1983), pp. 73–97.

J. A. Carrillo, S. Hittmeir, B. Volzone, and Y. Yao. "Nonlinear aggregation-diffusion equations: radial symmetry and long time asymptotics". *Inventiones Mathematicae* 218.3 (2019), pp. 889–977. arXiv: 1603.07767.

J. A. Carrillo, M. G. Delgadino, J. Dolbeault, R. L. Frank, and F. Hoffmann. "Reverse Hardy–Littlewood–Sobolev inequalities". *Journal des Mathematiques Pures et Appliquees* 132 (2019), pp. 133–165. arXiv: 1807.09189.

Bibliography II

J. A. Carrillo, M. G. Delgadino, R. L. Frank, and M. Lewin. "Fast Diffusion leads to partial mass concentration in Keller-Segel type stationary solutions". (2020), pp. 1–25. arXiv: 2012.08586.

J. A. Carrillo, D. G-C, Y. Yao, and C. Zeng. *Asymptotic simplification of Aggregation-Diffusion equations towards the heat kernel*. 2021. arXiv: 2105.13323.

J. Carrillo. "Entropy solutions for nonlinear degenerate problems". *Arch. Ration. Mech. Anal.* 147.4 (1999), pp. 269–361.

J. A. Cañizo, J. A. Carrillo, and M. E. Schonbek. "Decay rates for a class of diffusive-dominated interaction equations". *J. Math. Anal. Appl.* 389.1 (2012), pp. 541–557. arXiv: 1106.5880.

J. A. Carrillo, D. G-C, and J. L. Vázquez. *Infinite-time concentration in Aggregation–Diffusion equations with a given potential*. 2021. arXiv: 2103.12631.

M. G. Crandall, H. Ishii, and P.-L. Lions. "User's guide to viscosity solutions of second order Partial Differential Equation". *Bull. Am. Math. Soc.* 27.1 (1992), pp. 1–67.

C. Cao and X. Li. "Large Time Asymptotic Behaviors of Two Types of Fast Diffusion Equations". (2020). arXiv: 2011.02343.

Bibliography III

J. A. Carrillo and D. Slepčev. "Example of a displacement convex functional of first order". *Calculus of Variations and Partial Differential Equations* 36.4 (2009), pp. 547–564.

J. Dolbeault and B. Perthame. "Optimal critical mass in the two dimensional Keller–Segel model in". *Comptes Rendus Mathematique* 339.9 (Nov. 2004), pp. 611–616.

F. Demengel and R. Temam. "Convex functions of a measure and applications". *Indiana Univ. Math. J.* 33.5 (1986), pp. 673–709.

M. Giaquinta and S. Hildebrandt. *Calculus of variations. I.* Vol. 310. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. The Lagrangian formalism. Springer-Verlag, Berlin, 1996, pp. xxx+474.

M. A. Herrero and J. J. Velázquez. "Chemotactic collapse for the Keller-Segel model". *J. Math. Biol.* 35.2 (1996), pp. 177–194.

W. Jäger and S. Luckhaus. "On explosions of solutions to a system of partial differential equations modelling chemotaxis". *Transactions of the American Mathematical Society* 329.2 (1992), pp. 819–824.

- P.-E. Jabin and Z. Wang. "Mean Field Limit for Stochastic Particle Systems". *Active Particles, Volume 1*. Ed. by N. Bellomo, P. Degond, and E. Tadmor. Modeling and Simulation in Science, Engineering and Technology. Cham: Springer International Publishing, 2017, pp. 379–402.
- A. Kiselev, F. Nazarov, and A. Volberg. "Global well-posedness for the critical 2D dissipative quasi-geostrophic equation". *Invent. Math.* 167.3 (2007), pp. 445–453.
- S. N. Kružkov. "First Order Quasilinear Equations in Several Independent Variables". *Math. USSR-Sbornik* 10.2 (1970), pp. 217–243.
- E. H. Lieb. "Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation". *Stud. Appl. Math.* 57.2 (1977), pp. 93–105.
- E. H. Lieb. "Sharp Constants in the Hardy-Littlewood-Sobolev and Related Inequalities". *Ann. Math.* 118.2 (1983), p. 349.
- R. J. McCann. "A convexity principle for interacting gases". *Advances in Mathematics* 128.1 (1997), pp. 153–179.
- J. L. Vázquez. *The Porous Medium Equation*. Oxford University Press, 2006, pp. 1–648.

The heat equation and Brownian motion in $d = 1$

Go back

Consider an stochastic particle jumping over the mesh $\{\ldots, -h, 0, h, 2h, \ldots\}$ $(h > 0)$.

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$$\mathbb{P}(X_{n+1} = jh \mid X_n = ih) = \begin{cases} \frac{1}{2} & \text{if } |i - j| = 1, \\ 0 & \text{otherwise} \end{cases}$$

The heat equation and Brownian motion in $d = 1$

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This is a classical discretisation of the stochastic version of (HE): $\partial_t \rho = \frac{1}{2} \Delta \rho$.

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Go back

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$$dX_t = dW_t$$

Consider 1 particle. Using a similar arguments, for the stochastic equation

$$\mathrm{d}X_t = \underbrace{\mu(t, X_t)}_{\text{drift}}\,\mathrm{d}t + \underbrace{\sigma(t, X_t)\,\mathrm{d}W_t}_{\text{diffusion}}$$

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its probability density ρ is the solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t}(t, x) = -\operatorname{div}(\mu(t, x)\rho(t, x)) + \Delta(D(t, x)\rho(t, x))$$

where $D = \frac{\sigma^2}{2}$.

Imagine now we have N stochastic particles at positions $X_1(t), \cdots, X_N(t)$. We assume they have equal mass.

Recall the empirical measure $\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_j(t)}$

[2] Convergence in law: pointwise convergence of distribution functions at continuity points of the limit

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Recall the empirical measure $\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_j(t)}$

Assume the evolution of the particles is given by the system of SODEs

$$\mathrm{d}X_i = -\frac{1}{N}\sum_{j\neq i} \nabla W(X_i - X_j) - \frac{1}{N}\nabla V(X_i) + \sqrt{2D}\,\mathrm{d}W_t^i$$

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$$\partial_t \rho = \mathrm{div}(\rho \nabla(W * \rho + V)) + D\Delta\rho.$$

This corresponds to $U(\rho) = D\rho\log\rho$.

Mean-Field Approximation: As $N \to \infty$

$$\mu_0^N \to \rho_0 \text{ in the tight topology} \implies \mu_t^N \to \rho(t) \text{ in law for a.e. } t > 0.$$

For the details see, e.g., [Jabin and Wang 2017].

[2] Convergence in law: pointwise convergence of distribution functions at continuity points of the limit

PME: Finite time extinction

Go back

Let $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ and $d \geq 3$ and $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$:

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$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{q} \int_{\mathbb{R}^d} \rho^q \overset{\text{PDE}}{=} -C \int_{\mathbb{R}^d} |\nabla \rho^{\frac{m+q}{2}}|^2$$

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PME: Finite time extinction

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where $2^* = \frac{1}{2} - \frac{1}{d}$.

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where $2^* = \frac{1}{2} - \frac{1}{d}$.
The equation $\frac{\mathrm{d}}{\mathrm{d}t} X = -C X^\alpha$ where $\alpha < 1$ has finite time extinction.

PME: Asymptotic loss of mass

▸ Go back

Take $\varepsilon > 0$, taking $\rho_0^\varepsilon \in L^q$ with $\|\rho_0 - \rho_0^\varepsilon\|_{L^1} \le \varepsilon$. For $t \ge T_\varepsilon^*$

$$\|\rho(t)\|_{L^1} \le \|\rho(t) - \rho^\varepsilon(t)\|_{L^1} + \|\rho^\varepsilon(t)\|_{L^1}$$

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Computation of the Wasserstein gradient

Following [Ambrosio, Gigli, and Savare 2005, §10.4.1] Go back

$\mathcal{P}_2$ is not a vector space, so there we are not using the intrinsic notion of Fréchet gradient.
The correct notion is **Fréchet subdifferentials** (we will not define it here).
Also, we can see $\mathcal{P}_2$ inside the space of measures.
Fix $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then, the tangent space is given by

$$\mathrm{Tan}_{\rho_0} \mathcal{P}_2(\mathbb{R}^d) = \left\{ \xi : \exists \zeta_n \in C_c(\mathbb{R}^d, \mathbb{R}) \text{ s.t. } \int_{\mathbb{R}^d} |\xi - \nabla \zeta_n|^2 \, \mathrm{d}\rho_0 \to 0 \right\}$$

Take $\xi = \nabla \zeta$ with $\zeta \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Then, by [Ambrosio, Gigli, and Savare 2005, Lemma 5.5.3]

$$\rho_\varepsilon = (1_{\mathbb{R}^d} + \varepsilon \xi)_\# \rho_0 = \frac{\rho_0}{\det(1_{\mathbb{R}^d} + \varepsilon \nabla \xi)} \circ (1_{\mathbb{R}^d} + \varepsilon \xi)^{-1}$$

The map $(x, \varepsilon) \mapsto \rho_\varepsilon(x)$ is C^2 and

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = \rho_0, \qquad \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon = 0} \rho_\varepsilon = -\operatorname{div}(\rho \xi).$$

For ε small enough $1_{\mathbb{R}^d} + \varepsilon \nabla \zeta$ is an optimal transport map, so ρ_ε is a constant-speed geodesic.
Hence, using standard variation formulae (see [Giaquinta and Hildebrandt 1996])

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}[\rho_\varepsilon] - \mathcal{F}[\rho_0]}{\varepsilon} = -\int_{\mathbb{R}^d} \frac{\delta F}{\delta \rho}[\rho_0] \nabla \cdot (\rho \xi) = \int_{\mathbb{R}^d} \nabla \zeta \nabla \frac{\delta F}{\delta \rho}[\rho_0] \, \mathrm{d}\rho$$

This characterises $\nabla_{d_2} \mathcal{F} = -\operatorname{div}(\rho \nabla \frac{\delta F}{\delta \rho})$ in a broad distributional sense.

More on ∇_{L^2}

Go back

Let

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} F(x, \rho(x), \nabla \rho(x)) \, \mathrm{d}x.$$

More on ∇_{L^2}

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Let

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} F(x, \rho(x), \nabla \rho(x)) \, \mathrm{d}x.$$

Expanding $F(x, s, \xi)$ in Taylor expansion yields

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}[\rho_0 + \varepsilon \varphi] - \mathcal{F}[\rho_0]}{\varepsilon} = \int_{\mathbb{R}^d} \left(\frac{\partial F}{\partial s}(x, \rho_0, \nabla \rho_0) \varphi + \frac{\partial F}{\partial \xi}(x, \rho_0, \nabla \rho_0) \cdot \nabla \varphi \right)$$

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\end{aligned}$$

Thus

$$\nabla_{L^2} \mathcal{F}[\rho_0] = \frac{\delta \mathcal{F}}{\delta \rho}[\rho_0] = \frac{\partial F}{\partial s}[\rho_0] - \operatorname{div} \left(\frac{\partial F}{\partial \xi}[\rho_0] \right).$$

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This is the Euler-Lagrange equation!

Convex functions of a measure

Following [Demengel and Temam 1986] Go back

Given

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} f(\rho)\, \mathrm{d}x$$

where $f : \mathbb{R} \to \mathbb{R}$.

The questions is what is the natural lower semicontinuous extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$ with the weak-$\star$ topology.

Given a measure μ and mollifiers η_ε we define $\rho_\varepsilon = \mu * \eta_\varepsilon$.
For $|f(\xi)| \le C(1 + |\xi|)$ define

$$f_\infty(\xi) = \lim_{t \to \infty} \frac{f(t\xi)}{t}.$$

Since we can use the Lebesgue decomposition theorem $\mu = \rho\, \mathrm{d}x + \mu^s$, where ρ is the Radon-Nikodym derivative of μ. Then

$$\widetilde{F}[\mu] = \int_{\mathbb{R}^d} f(\rho)\, \mathrm{d}x + f_\infty(\mu^s).$$

The notion of $f_\infty(\mu^s)$ is tricky (but possible) to define.
If $f(s) = s^m$ with $m < 1$, then $f_\infty = 0$.

Curves of maximal slope

(see [Ambrosio, Gigli, and Savare 2005]) ▸ Go back

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Typically, $\frac{\partial \rho}{\partial t} = -\nabla_X \mathcal{F}[\rho(t)]$ for $X = L^2, H^1$ is satisfied in the dual sense.

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The way in which $\dfrac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)]$ in rather tricky since $\mathcal{P}_2$ is not linear a space.

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The way in which $\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)]$ in rather tricky since $\mathcal{P}_2$ is not linear a space.

The main idea is the equivalence for $u : [0,T] \to \mathbb{R}^d$ that

$$u'(t) = -\nabla \mathcal{F}(u), \qquad \Longleftrightarrow \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F} \circ u) = -|\nabla F(u)||u'| & \text{orientation} \\ |u'| = |\nabla \mathcal{F}(u)| & \text{norm} \end{cases}$$

Curves of maximal slope

(see [Ambrosio, Gigli, and Savare 2005]) Go back

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The way in which $\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)]$ in rather tricky since $\mathcal{P}_2$ is not linear a space.

The main idea is the equivalence for $u : [0, T] \to \mathbb{R}^d$ that

$$u'(t) = -\nabla \mathcal{F}(u), \qquad \Longleftrightarrow \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F} \circ u) = -|\nabla F(u)||u'| & \text{orientation} \\ |u'| = |\nabla \mathcal{F}(u)| & \text{norm} \end{cases}$$

We define the metric slopes

$$|\mu'|(t) = \limsup_{h \to 0} \frac{d_2(\mu(t+h), \mu(t))}{h}, \qquad |\partial \mathcal{F}|[\mu] = \limsup_{\nu \to \mu} \frac{(\mathcal{F}[\mu] - \mathcal{F}[\nu])_+}{d_2(\mu, \nu)}$$

Curves of maximal slope

(see [Ambrosio, Gigli, and Savare 2005]) Go back

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Definition 1 Maximal slope curve

A locally abs. cont. curve $t \mapsto \mu(t) \in \mathcal{P}_2(\mathbb{R}^d)$ such that $t \mapsto \mathcal{F}[\mu(t)]$ is abs. cont. and

$$\frac{1}{2} \int_s^t |\mu'|^2(r)\,\mathrm{d}r + \frac{1}{2} \int_s^t |\partial \mathcal{F}|^2[\mu(r)]\,\mathrm{d}r \le \mathcal{F}[\mu(s)] - \mathcal{F}[\mu(t)] \qquad \forall 0 \le s < t \le T$$

Lieb's trick

Go back

Given a radially decreasing $\rho \geq 0$, $\rho^q \in L^1(B_R)$ for some $q > 0$ (for any $R \leq \infty$), using and old trick of Lieb's (see [Lieb 1977; Lieb 1983]) we get, for $|x| \leq R$,

$$\int_{B_R} \rho^q \,\mathrm{d}x = n\omega_n \int_0^R \rho(r)^q r^{n-1} \,\mathrm{d}r \geq n\omega_n \int_0^{|x|} \rho(r)^q r^{n-1} \,\mathrm{d}r \geq n\omega_n \rho(x)^q \int_0^{|x|} r^{n-1} \,\mathrm{d}r.$$

Hence, we deduce the point-wise estimate

$$\rho(x) \leq \left(\frac{\int_{B_R} \rho^q}{n\omega_n |x|^n} \right)^{\frac{1}{q}}. \tag{1}$$

It is easy to see that (1) is not sharp. However, it is useful to prove tightness for sets of probability measures.

For (DE) **entropy solutions**:

$$\rho_0 \in L^1 \implies \exists! \rho \in C([0, +\infty); L^1(\mathbb{R}^d))$$

(see, e.g. [Kružkov 1970; Carrillo 1999])

Limitations of the L^1 framework

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The L^1 framework is not enough!

Distances between measures

Total variation: $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^d)$.

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We want to construct a distance between measures such that

$$d(\delta_a, \delta_b) = |a - b|.$$

The Wasserstein distance

▸ Go back

Given $\mu, \nu \in \mathcal{P}(X)$, taking plans between μ and ν:

$$\Pi(\mu,\nu) = \Big\{\pi \in \mathcal{P}(X \times Y) : \quad \pi(A \times Y) = \mu(A), \quad \pi(X \times B) = \nu(B)\Big\}.$$

we define the p-Wasserstein distance

$$d_p(\mu,\nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} |x-y|^p \, \mathrm{d}\pi(x,y) \right)^{\frac{1}{p}}$$

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The correct space to work with this distance is

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \, \mathrm{d}\mu(x) \right\}$$

We endow $\mathcal{P}_p(\mathbb{R}^d)$ with the distance d_p.

Sketch of the proof of [Carrillo, G-C, and Vázquez 2021]

Go back

For simplicity, we restrict to first bounded domains

$$\begin{cases} \partial_t u = \Delta \rho^m + \operatorname{div}(u \nabla V) & x \in \Omega, t > 0 \\ \partial_n u = \partial_n V = 0 & \partial\Omega. \end{cases}$$

For $m \in (0,1)$, $V \in W^{1,\infty}$, then the solution is well-defined and regular for all $t > 0$.

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If ρ is radially symmetric, the mass function

$$M(t,r) = \int_{B_r} \rho(t,x)\, \mathrm{d}x$$

can be studied as viscosity solution of a Hamilton-Jacobi type equation
(see [Crandall, Ishii, and Lions 1992]).

For viscosity solutions may properties are known: stability, C^α regularity, ...

We construct an explicit initial datum ρ_0 such that $\partial_t M \geq 0$.

In the limit $M(t,\cdot) \nearrow M_\infty$, a solution of the mass of Euler-Lagrange and $M_\infty(R_v) = 1$.

There are no solutions of sufficient mass. Therefore $M_\infty(0^+) = a > 0$.

Sketch of proof of [Carrillo, G-C, Yao, and Zeng 2021]

Go back

First, we prove well-posedness by Duhamel's formula and that, in rescaled variable

- If $\nabla W \in L^n(\mathbb{R}^d)$ then $\sup_{\tau \geq 1} \|\widetilde{\rho}(\tau, \cdot)\|_{H^1} < \infty$.
- If $n \geq 2$, $\nabla W \in L^n(\mathbb{R}^d)$ and $\Delta W \in L^{\frac{n}{2}}(\mathbb{R}^d)$ then $\sup_{\tau \geq 1} \|\widetilde{\rho}(\tau, \cdot)\|_{C^\alpha} < \infty$ (modulus of continuity arguments, e.g. [Kiselev, Nazarov, and Volberg 2007])

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Like for (HE), we can use logarithmic Sobolev inequality
to recover an Ordinary Differential Inequality for E_1
(where the terms from W are a controlled errors)

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Lastly, we apply the Csiszar-Kullback inequality

$$\|\rho(t, \cdot) - K(t, \cdot)\|_{L^1} = \|\widetilde{\rho}(t, \cdot) - G(\cdot)\|_{L^1}$$

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$$\leq 2\sqrt{E_1(\widetilde{\rho} \| G)}$$

Sketch of proof of [Carrillo, G-C, Yao, and Zeng 2021]

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$$\to 0, \qquad \text{as } t \to \infty.$$